Journal of Statistical Physics, Vol. 127, No. 4, May 2007 (© 2007)
DOI: 10.1007/s10955-006-9269-6

Global Solution to the Relativistic Enskog Equation with Near-Vacuum Data

Zhenglu Jiang[1]

Received July 4, 2006; accepted December 10, 2006
Published Online: March 23, 2007

We give two hypotheses of the relativistic collision kernal and show the existence and uniqueness of the global mild solution to the relativistic Enskog equation with the initial data near the vacuum for a hard sphere gas.

KEY WORDS: Relativistic, Enskog equation, global solution

2000 Mathematics Subject Classification. 76P05; 35Q75; 82-02.

1. INTRODUCTION

We will show the existence and uniqueness of the global mild solution to the relativistic Enskog equation with near-vacuum initial data for a hard sphere gas. By using a similar derivation as for the relativistic Boltzmann equation,[4] the relativistic Enskog equation can be obtained as follows (see Ref. 12):

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{p_0}\frac{\partial f}{\partial \mathbf{x}} = Q(f) \tag{1.1}$$

where $t \in [0, +\infty)$, $\mathbf{x} \in \mathbf{R}^3$, $\mathbf{p} \in \mathbf{R}^3$, $p_0 = (1 + |\mathbf{p}|^2)^{1/2}$ and $Q(f) = Q(f)(t, \mathbf{x}, \mathbf{p})$ is the relativistic Enskog collision operator which is expressed by the difference between the gain and loss terms respectively defined by

$$Q^+(f)(t, \mathbf{x}, \mathbf{p}) = \frac{a^2}{p_0}\int_{\mathbf{R}^3}\frac{d^3\mathbf{p}_1}{p_{10}}\int_{S_+^2} d\omega F^+(f) f(t, \mathbf{x}, \mathbf{p}') f(t, \mathbf{x} + a\omega, \mathbf{p}_1') B(g, \theta) \tag{1.2}$$

$$Q^-(f)(t, \mathbf{x}, \mathbf{p}) = \frac{a^2}{p_0}\int_{\mathbf{R}^3}\frac{d^3\mathbf{p}_1}{p_{10}}\int_{S_+^2} d\omega F^-(f) f(t, \mathbf{x}, \mathbf{p}) f(t, \mathbf{x} - a\omega, \mathbf{p}_1) B(g, \theta) \tag{1.3}$$

[1] Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China; e-mail: mcsjzl@mail.sysu.edu.cn

where $f = f(t, \mathbf{x}, \mathbf{p})$ is a distribution function of a one-particle classical relativistic gas without external forces, a is a diameter of hard sphere ($a > 0$) and $F^{\pm}$ are two functionals on a Banach space M defined in Sec. 2. More precisely speaking, $F^{\pm}$ are high-density collision frequencies which are physically defined by a geometrical factor Y that depends on the space density $\rho(t, \mathbf{x}) = \int_{\mathbf{R}^3} f(t, \mathbf{x}, \mathbf{p})\, d^3\mathbf{p}$ at the time t and the point of contact [e.g. $F^+ = Y(\rho(t, \mathbf{x} + a\omega/2))$ and $F^- = Y(\rho(t, \mathbf{x} - a\omega/2))$]. The other different parts in Eqs. (1.2) and (1.3) are explained below.

$\mathbf{p}$ and $\mathbf{p}_1$ are dimensionless momenta of two colliding particles immediately before collision while $\mathbf{p}'$ and $\mathbf{p}_1'$ are dimensionless momenta after collision; $p_0 = (1 + |\mathbf{p}|^2)^{1/2}$ and $p_{10} = (1 + |\mathbf{p}_1|^2)^{1/2}$ are the dimensionless energy of two colliding particles immediately before collision while $p_0' = (1 + |\mathbf{p}'|^2)^{1/2}$ and $p_{10}' = (1 + |\mathbf{p}_1'|^2)^{1/2}$ are the dimensionless energy after collision. $\mathbf{R}^3$ is a three-dimensional Euclidean space and $S_+^2 = \{\omega \in S^2 : \omega(\mathbf{p}/p_0 - \mathbf{p}_1/p_{10}) \geq 0\}$ a subset of a unit sphere surface S^2. $B(g, \theta)$ is given by $B(g, \theta) = gs^{1/2}\sigma(g, \theta)/2$, where $\sigma(g, \theta)$ is a scattering cross section, $s = |p_{10} + p_0|^2 - |\mathbf{p}_1 + \mathbf{p}|^2$, $g = \sqrt{|p_{10} - p_0|^2 - |\mathbf{p}_1 - \mathbf{p}|^2}/2$, θ is the scattering angle defined in $[0, \pi]$ by $\cos\theta = 1 - 2[(p_0 - p_{10})(p_0 - p_0') - (\mathbf{p} - \mathbf{p}_1)(\mathbf{p} - \mathbf{p}')]/(4 - s)$. Obviously, $s = 4 + 4g^2$. $\omega = (\cos\psi \sin\theta, \sin\psi \sin\theta, \cos\theta)$ varies on S_+^2, where $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$. Put $\mathbf{p}' = \mathbf{p} + q\omega$ and $\mathbf{p}_1' = \mathbf{p} - q\omega$. Then, by using a similar derivation as given by Glassey and Strauss[11] for the relativistic Boltzmann equation, we have

$$q \equiv q(\mathbf{p}, \mathbf{p}_1, \omega) = \frac{2(p_0 + p_{10})p_0 p_{10}[\omega(\mathbf{p}_1/p_{10} - \mathbf{p}/p_0)]}{(p_0 + p_{10})^2 - [\omega(\mathbf{p} + \mathbf{p}_1)]^2}, \tag{1.4}$$

$$g/s^{1/2} = 8\frac{(p_0 + p_{10})^2|\omega(\mathbf{p}_1/p_{10} - \mathbf{p}/p_0)|}{\{(p_0 + p_{10})^2 - [\omega(\mathbf{p} + \mathbf{p}_1)]^2\}^2}. \tag{1.5}$$

Here the scattering angle θ can be regarded as a function of the variables $\mathbf{p}$, $\mathbf{p}_1$ and ω, i.e., $\theta \equiv \theta(\mathbf{p}, \mathbf{p}_1, \omega)$. $\mathbf{p}'$ and $\mathbf{p}_1'$ are bounded for bounded pre-collisional momenta and lie on an ellipsoid when they are plotted in a plane, see detail explanation in Ref. 1.

The Moller velocity is defined as $v_M = gs^{1/2}/(p_0 p_{10})$, thus it can be found that $v_M^2 = |\mathbf{p}/p_0 - \mathbf{p}_1/p_{10}|^2 - |\mathbf{p} \times \mathbf{p}_1/(p_0 p_{10})|^2$ and that $v_M \leq |\mathbf{p}/p_0 - \mathbf{p}_1/p_{10}|$.

As a comparison the relativistic Boltzmann equation is the relativistic Enskog equation with the factor $a^2 F^{\pm}$ constant and the diameter a equal to zero in the density variables. Boltzmann's equation provides a successful description for dilute gases and is no longer valid when the density of the gas increases. The Enskog equation proposed by Enskog[8] in 1922 is a modification of the Boltzmann equation to explain the dynamical behavior of the density profile of a moderately dense gas. It is thus a suitable idea that the relativistic Enskog equation is used to model a hard sphere relativistic gas.

There are many results about the relativistic Boltzmann equation, such as global existence proof of Dudyński and Ekiel-Jeżewska[6,7] and properties of the relativistic collision operator given by Glassey and Strauss,[10] and background information of the classical Enskog equation may be found in Refs. 2, 8, 14. An existence and uniqueness theorem has been given by Galeano et al.[12] for the global solution to the relativistic Enskog equation with data near the vacuum for a hard sphere gas. In their theorem (see Theorem 3.2 in Ref. 12), a set M_R is defined by $M_R = \{f \in M : ||f|| \leq R\}$ with

$$R^2 < \beta^4|v|/(16\pi^2 cLa) \tag{1.6}$$

and an initial datum f_0 satisfies

$$||f_0|| < Re^{\beta|x|^2}/2, \tag{1.7}$$

where M is defined by

$$M = \left\{ f \in C([0,\infty) \times \mathbf{R}^3 \times \mathbf{R}^3) : \quad \begin{array}{l} \text{there exists } c > 0 \text{ such that} \\ |f(t,x,v)| \leq ce^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)} \end{array} \right\}$$

with a norm

$$||f|| = \sup_{t,x,v} \left\{ e^{\beta(\sqrt{1+|v|^2}+|x+tv|^2)} |f(t,x,v)| \right\},$$

c, L, a and β are positive constants, t is a time variable in $[0,\infty)$, x and v are space and momentum variables in $\mathbf{R}^3$ respectively. These assumptions imply that M_R is an empty set and that none of the initial datum f_0 occurs. Below let us prove this claim. We first prove that M_R is an empty set. Assume that M_R is not an empty set. Then there exists a function f in M_R such that $||f|| \leq R$. Thus $R \geq 0$. By assumption (1.6), we know that

$$R < \sqrt{\beta^4|v|/(16\pi^2 cLa)}. \tag{1.8}$$

Since v is a momentum variables in $\mathbf{R}^3$, by setting $v = 0$, (1.8) shows that $R < 0$ as $v = 0$. This is in contradiction with $R \geq 0$. Hence M_R is an empty set. Next, we show that none of the initial datum f_0 occurs. Assume that there exists a function f_0 satisfying (1.7). It can be known from (1.7) that $R > 0$. By (1.6), (1.8) then follows. Since v is a momentum variables in $\mathbf{R}^3$, by letting $v = 0$, (1.8) shows that $R < 0$ as $v = 0$. This is in contradiction with $R > 0$. None of the initial datum f_0 hence occurs. If (1.6) and (1.7) are replaced with $R^2 \leq /\beta^4|v|/(16\pi^2 cLa)$ and $||f_0|| \leq Re^{\beta|x|^2}/2$ respectively, then $M_R = \{0\}$ and $f_0 = 0$. Thus one can deduce that a unique solution to the relativistic Enskog equation in their theorem is in fact zero. Hence this result is also trivial. Notice that v is a variable in $\mathbf{R}^3$ and that (1.6) holds for all v in $\mathbf{R}^3$. If $|v|$ in (1.6) is replaced with a positive constant v_0, that is, (1.6) is changed as $R^2 < \beta^4 v_0/(16\pi^2 cLa)$, then the problem is non-trivial.

Recently, global existence of mild solutions has been proved by Glassey[9] for the relativistic Boltzmann equation with near-vacuum data and many relevant papers of both classical and relativistic cases can be found in the reference. Now there is not yet this result for the relativistic Enskog equation. The aim of this paper is to extend this result into the case of the relativistic Enskog equation. In Sec. 2 two hypotheses of the relativistic collision kernel are given and a Banach space and its operators are constructed. Then an existence and uniqueness theorem of global mild solution to the relativistic Enskog equation with near-vacuum data is given in Sec. 3.

2. HYPOTHESES AND OPERATORS

Let us begin by assuming that there is a positive function $m(\mathbf{x}, \mathbf{p})$ such that a non-negative function $B(g, \theta) = gs^{1/2}\sigma(g, \theta)/2$ satisfies two hypotheses as follows:

$$\frac{1}{p_0}\int_0^t d\tau \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S_+^2} d\omega m(\mathbf{x} + \tau\mathbf{p}/p_0 - \tau\mathbf{p}'/p_0', \mathbf{p}') \times m(\mathbf{x} + a\omega + \tau\mathbf{p}/p_0 - \tau\mathbf{p}_1'/p_{10}', \mathbf{p}_1')B(g, \theta) \le m(\mathbf{x}, \mathbf{p})K, \tag{2.1}$$

$$\frac{1}{p_0}\int_0^t d\tau \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S_+^2} d\omega m(\mathbf{x} - a\omega + \tau\mathbf{p}/p_0 - \tau\mathbf{p}_1/p_{10}, \mathbf{p}_1)B(g, \theta) \le K. \tag{2.2}$$

for any $\mathbf{x} \in \mathbf{R}^3$, $\mathbf{p} \in \mathbf{R}^3$, $t \in \mathbf{R}^+$ and some positive constant K. It can be known from the recent work of Glassey[9] that there exist two such functions $m(\mathbf{x}, \mathbf{p})$ and $B(g, \theta)$ satisfying (2.1) and (2.2). For exampe, as given by Glassey,[9] we assume that

$$m(\mathbf{x}, \mathbf{p}) = (1 + |\mathbf{x} \times \mathbf{p}|)^{-(1+\delta)/2} e^{-p_0}, \tag{2.3}$$

$$\sigma \equiv \sigma(\mathbf{p}, \mathbf{p}_1, \omega) = |\omega(\mathbf{p}_1 \times \mathbf{p})|\tilde{\sigma}(\omega)/[p_{10}g(1 + g^2)^{\delta+1/2}], \tag{2.4}$$

for any fixed $\delta \in (0, 1)$, where $\tilde{\sigma}(\omega)$ is a nonnegative, bounded and continuous function such that $\int_{S_+^2} \tilde{\sigma}(\omega)/(1 + |\mathbf{z}\omega|)\, d\omega \le c_0|\mathbf{z}|^{-1}$ for some positive constant c_0 and every nonzero element $\mathbf{z} \in \mathbf{R}^3$. Thus a similar integral estimate to that developed by Glassey[9] leads to the fact that (2.1) and (2.2) hold if $m(\mathbf{x}, \mathbf{p})$ and $B(g, \theta)$ are defined by (2.3) and (2.4) respectively. This indicates that our assumptions (2.1) and (2.2) are valid for the relativistic Enskog equation.

Then we can construct a subset M of a Banach space $C([0, \infty) \times \mathbf{R}^3 \times \mathbf{R}^3)$, which has the property that every element $f = f(t, \mathbf{x}, \mathbf{p}) \in M$ if and only if there exists a positive constant c such that f satisfies $|f^\#(t, \mathbf{x}, \mathbf{p})| \le cm(\mathbf{x}, \mathbf{p})$, where and below everywhere, $f^\#$ is expressed as $f^\#(t, \mathbf{x}, \mathbf{p}) = f(t, \mathbf{x} + t\mathbf{p}/p_0, \mathbf{p})$ for any measurable function f on $(0, +\infty) \times \mathbf{R}^3 \times \mathbf{R}^3$. It follows that M is a Banach space when it has a norm $||f|| = \sup_{t,\mathbf{x},\mathbf{p}}\{|f^{\#}(t, \mathbf{x}, \mathbf{p})|m^{-1}(\mathbf{x}, \mathbf{p})\}$. This space will be used below.

The relativistic Enskog equation (1.1) can be also written as

$$\frac{d}{dt} f^{\#}(t, \mathbf{x}, \mathbf{p}) = Q(f)^{\#}(t, \mathbf{x}, \mathbf{p}),$$

which leads to the following integral equation

$$f^{\#}(t, \mathbf{x}, \mathbf{p}) = f_0(\mathbf{x}, \mathbf{p}) + \int_0^t Q(f)^{\#}(\tau, \mathbf{x}, \mathbf{p})\, d\tau. \tag{2.5}$$

A function $f(t, \mathbf{x}, \mathbf{p})$ is called global mild solution to the Enskog equation (1.1) if $f(t, \mathbf{x}, \mathbf{p})$ satisfies the above integral equation (2.5) for almost every $(t, \mathbf{x}, \mathbf{p}) \in [0, +\infty) \times \mathbf{R}^3 \times \mathbf{R}^3$. The definition of the term "mild solution" also appears in the famous work of DiPerna and Lions[5] where they show a global existence proof for the classical Boltzmann equation.

By (1.2) and (1.3), $Q(f)^{\#}(t, \mathbf{x}, \mathbf{p})$ can be rewritten as the difference between the gain and loss terms of two other forms

$$Q^{+}(f)^{\#}(t, \mathbf{x}, \mathbf{p}) = \frac{a^2}{p_0} \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S^2_+} d\omega F^{+}(f) f^{\#}(t, \mathbf{x} + t\mathbf{p}/p_0 - t\mathbf{p}'/p'_0, \mathbf{p}')$$
$$\times f^{\#}(t, \mathbf{x}, +a\omega + t\mathbf{p}/p_0 - t\mathbf{p}'_1/p'_{10}, \mathbf{p}'_1) B(g, \theta), \tag{2.6}$$

$$Q^{-}(f)^{\#}(t, \mathbf{x}, \mathbf{p}) = \frac{a^2}{p_0} \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S^2_+} d\omega F^{-}(f) f^{\#}(t, \mathbf{x}, \mathbf{p})$$
$$\times f^{\#}(t, \mathbf{x} - a\omega + t\mathbf{p}/p_0 - t\mathbf{p}_1/p_{10}, \mathbf{p}_1) B(g, \theta). \tag{2.7}$$

According to (2.6) and (2.7), we can finally build a Banach space $\widetilde{M}$ defined by $\widetilde{M} = \{f^{\#} : f \in M\}$ with a norm $|||f^{\#}||| = ||f||$ and an operator J on $\widetilde{M}$ by

$$J(f^{\#}) = f_0(\mathbf{x}, \mathbf{p}) + \int_0^t Q(f)^{\#}(\tau, \mathbf{x}, \mathbf{p})\, d\tau, \tag{2.8}$$

since $F^{\pm}$ can be in fact regarded as two functionals on $\widetilde{M}$.

3. EXISTENCE AND UNIQUENESS

Let M_R be denoted by $M_R = \{f \in M : ||f|| \leq R\}$ for any $R \in \mathbf{R}_+$, where M is given in Sec. 2. We first have the following lemma:

Lemma 3.1. *Suppose that the conditions (2.1) and (2.2) hold and that $F^{\pm}$ are two functionals on M_R such that $|F^{\pm}(f) - F^{\pm}(g)| \leq L(R)||f - g||$ for any*

$f, g \in M_R$ where $L(R)$ is a positive nondecreasing function on $\mathbf{R}_+$. Then

$$\int_0^t |Q^+(f)^\#(\tau, \mathbf{x}, \mathbf{p})| d\tau \le C(R) m(\mathbf{x}, \mathbf{p}) ||f||^2,$$

$$\int_0^t |Q^-(f)^\#(\tau, \mathbf{x}, \mathbf{p})| d\tau \le C(R) m(\mathbf{x}, \mathbf{p}) ||f||^2$$

for any $f \in M_R$, where $C(R)$ is a positive nondecreasing function on $\mathbf{R}_+$.

Proof: It can be first found from the assumption of the two functionals $F^\pm$ that there exists a positive constant $\widetilde{L}(R) = L(R)R + |F^+(0)| + |F^-(0)|$ such that $|F^\pm(f)| \le \widetilde{L}(R)$ for any $f \in M_R$. It follows from (2.6) and (2.7) that

$$\int_0^t Q^+(f)^\#(\tau, \mathbf{x}, \mathbf{p})\, d\tau \le \frac{\widetilde{L}(R) a^2}{p_0} \int_0^t d\tau \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S_+^2} d\omega ||f||^2 \tag{3.1}$$
$$\times\, m(\mathbf{x} + \tau\mathbf{p}/p_0 - \tau\mathbf{p}'/p_0', \mathbf{p}') m(\mathbf{x} + a\omega + \tau\mathbf{p}/p_0 - \tau\mathbf{p}'_1/p_{10}', p_1') B(g, \theta),$$

$$\int_0^t Q^-(f)^\#(t, \mathbf{x}, \mathbf{p})\, d\tau \le \frac{\widetilde{L}(R) a^2}{p_0} \int_0^t d\tau \int_{\mathbf{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S_+^2} d\omega ||f||^2$$
$$\times\, m(\mathbf{x}, \mathbf{p}) m(\mathbf{x} - a\omega + \tau\mathbf{p}/p_0 - \tau\mathbf{p}_1/p_{10}, \mathbf{p}_1) B(g, \theta). \tag{3.2}$$

By (2.1) and (2.2), (3.1) and (3.2) give

$$\int_0^t Q^+(f)^\#(\tau, \mathbf{x}, \mathbf{p})\, d\tau \le \widetilde{L}(R) a^2 K m(\mathbf{x}, \mathbf{p}) ||f||^2,$$

$$\int_0^t Q^-(f)^\#(\tau, \mathbf{x}, \mathbf{p})\, d\tau \le \widetilde{L}(R) a^2 K m(\mathbf{x}, \mathbf{p}) ||f||^2.$$

Take $C(R) = \widetilde{L}(R) a^2 K$. It follows obviously that Lemma 3.1 holds. □

Then we can get the following theorem:

Theorem 3.2. *Suppose that the conditions (2.1) and (2.2) hold and that $F^\pm$ are two functionals on M_R such that $|F^\pm(f) - F^\pm(g)| \le L(R)||f - g||$ for any f, $g \in M_R$ where $L(R)$ is a positive nondecreasing function on R_+. Then there exists a positive constant R_0 such that the relativistic Enskog equation (1.1) has a unique non-negative global mild solution $f = f(t, \mathbf{x}, \mathbf{p}) \in M_{R_0}$ through a non-negative initial data $f_0 = f_0(\mathbf{x}, \mathbf{p})$ when $\sup_{\mathbf{x},\mathbf{p}}\{f_0(\mathbf{x}, \mathbf{p}) m^{-1}(\mathbf{x}, \mathbf{p})\}$ is sufficiently small.*

Theorem 3.2 shows that there exists a unique global mild solution to the relativistic Enskog Eq. (1.1) with the initial data near vacuum if a suitable assumption of the scattering kernel is given. Below is our proof of Theorem 3.2.

Proof: We first define a set $\widetilde{M}_R$ by $\widetilde{M}_R = \{f^\# : |||f^\#||| \leq R, f^\# \in \widetilde{M}|$, where $\widetilde{M}$ is given in Sec. 2. By (2.8), we have

$$|J(f^\#)|m^{-1}(\mathbf{x}, \mathbf{p}) \leq |f_0(\mathbf{x}, \mathbf{p})|m^{-1}(\mathbf{x}, \mathbf{p}) + 2C(R)||f||^2 \leq R/2 + 2C(R)R^2 \tag{3.3}$$

for any $f^\# \in \widetilde{M}_R$ and f_0 with $||f_0|| \leq R/2$. Since $C(R)$ is a positive nondescreasing function on $\mathbf{R}_+$, it follows that $|||J(f)^\#||| \leq R$ for sufficiently small R. Therefore J is an operator from $\widetilde{M}_R$ to itself for sufficiently small R. Similarly, it can be also found that J is a contractive operator on $\widetilde{M}_R$ for some suitably small R. Thus there exists a unique element $f^\# \in \widetilde{M}_R$ such that $f^\# = J(f^\#)$, i.e., (2.5) holds. It then follows from the same argument as the one in Ref. 3 (or see Refs. 9, 13, 15) that if $f_0(\mathbf{x}, \mathbf{p}) \geq 0$ then $f(t, \mathbf{x}, \mathbf{p}) \geq 0$. Hence the proof of Theorem 3.2 is finished. □

ACKNOWLEDGMENTS

The author is supported by NSFC 10271121, 10511120278 and 10611120371, and sponsored by SRF for ROCS, SEM. The author would like to thank the referees of this paper for their valuable comments on this work.

REFERENCES

1. H. Andreasson, S. Calogero and R. Illner, On blowup for gain-term-only classical and relativistic Boltzmann equations. *Math. Meth. Appl. Sci.* **27**:2231–2240 (2004).
2. L. Arkeryd, On the Enskog equation with large initial data. *SIAM J. Math. Anal.* **21**:631–646 (1990).
3. K. Bichteler, On the Cauchy problem of the relativistic Boltzmann equation. *Commun. Math. Phys.* **4**:352–364 (1967).
4. S. R. de Groot, W. A. Van Leeuwen and Ch. G. Van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
5. R. J. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability. *Ann. Math.* **130**:321 (1989).
6. M. Dudyński and M. L. Ekiel-Jeżewska, On the linearized relativistic Boltzmann equation. *Commun. Math. Phys.* **115**:607–629 (1988).
7. M. Dudyński and M. L. Ekiel-Jeżewska, Global existence proof for relativistic Boltzmann equation. *J. Stat. Phys.* **66**(3/4) (1992).
8. D. Enskog, Kinetiche Theorie der Wàrmeleitung, Reibung und Selbstdiffusion in gewissen werdichteten Gasen und Flubigkeiten. *Kungl. Sv. Vetenskapsakademiens Handl.* **63**:3–44 (1922), English Transl. in Brush, S. G., Kinetic Theory, vol 3 (Pergamon, New York, 1972).
9. R. Glassey, Global solutions to the Cauchy problem for the relativistic Boltzmann equation with near-vacuum data. *Comm. Math. Phys.* **26**:705–724 (2006).
10. R. Glassey and W. Strauss, On the derivatives of the collision map of relativistic particles. *Transp. Theory Stat. Phys.* **20**:55–68 (1991).
11. R. Glassey and W. Strauss, Asymptotic stability of the relativistic Maxwellian via fourteen moments. *Transp. Theory Stat. Phys.* **24**:657–678 (1995).

12. R. Galeano, O. Vasquez and B. Orozco, The relativistic Enskog equation. *J. Differ. Equ. Conf.* **13**:21–27 (2005).
13. R. Illner and M. Shinbrot, The Boltzmann equation, global existence for a rare gas in an infinite vacuum. *Comm. Math. Phys.* **95**:217–226 (1984).
14. J. Polewczak, Global existence and asymptotic behavior for the nonlinear Enskog equation. *SIAM J. Appl. Math.* **49**:952–959 (1989).
15. S. Ukai, Solutions of the Boltzmann equation. *Stud. Math. Appl.* **18**:37–96 (1986).